

# AN OVERLAPPING GENERATIONS MODEL OF EXHAUSTIBLE NATURAL RESOURCES

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## Introduction

The purpose of this paper is to extend the recent Balasko and Shell [2] existence and welfare results on equilibrium in overlapping generations models to an economy with an exhaustible natural resource. Their paper considers an economy consisting of an infinite sequence of two-period lived agents each possessing a strictly positive endowment of a finite number of perishable commodities in each period of life, and demonstrates that equilibrium exists for a particular class of agents' preference relations, but that equilibrium allocations may fail to be Pareto optimal. We shall examine a similar overlapping generations economy where in addition to perishable commodities agents may consume, or costlessly store, an essential exhaustible resource available in fixed finite supply only at the inception of economic activity and demonstrate conditions under which equilibrium exists, the resource is never exhausted and every equilibrium allocation is fully Pareto optimal. The postulated conditions are not the weakest for which the result is valid but yield a remarkably simple proof of the central theorem.

Following Balasko and Shell, we examine a simplified economy composed of two-period lived agents indexed by their dates of birth  $t = 0, 1, 2, \dots$ . Agents of generation zero live out the balance of their lives in period one. Consumption of agent  $t$  in period  $s$  is  $(x_t^s, y_t^s)$  where  $x_t^s \in R_t^+$  denotes consumption of perishable commodities and  $y_t^s \in R_+^1$  consumption of the exhaustible resource. The endowment of commodities to agent  $t$  in period  $s$  is  $\omega_t^s \in R_+^k$ . Only agents present at the inception of the economy are endowed with a stock of the resource and this endowment is  $S_0 \in (0, \infty)$ . For simplicity we shall carry out the analysis as if each generation consists of a single agent.

Each agent derives utility from consuming goods during his lifetime and the utility function of agent  $t$  is  $u_t(x_t)$   $t = 0, 1, 2, \dots$  where

$$\begin{aligned} x_0 &= (x_0^1, y_0^1) \text{ for } t = 0 \\ x_t &= (x_t^t, y_t^t; x_t^{t+1}, y_t^{t+1}) \text{ for } t = 1, 2, \dots \end{aligned}$$

Similarly, the endowment of agent  $t$  may be denoted

$$\begin{aligned} \omega_0 &= (\omega_0^1; S_0) \text{ for } t = 0 \\ \omega_t &= (\omega_t^t; \omega_t^{t+1}) \text{ for } t > 0 \end{aligned}$$

Agents choose their lifetime consumption profiles to solve the constrained utility-maximization problem

$$\begin{aligned} (1.1) \\ \text{MAX } u_0(x_0) \quad &\text{s.t. } p^1 x_0^1 \leq p^1 \omega_0^1 + q^1 S_0^1 \\ &S_0^1 = S_0 - y_0^1 \geq 0 \text{ for } t = 0 \\ \text{MAX } u_t(x_t) \quad &\text{s.t. } p^t x_t^t + p^{t+1} x_t^{t+1} \leq p^t \omega_t^t + p^{t+1} \omega_t^{t+1} - q^t S_t^t + q^{t+1} S_t^{t+1} \\ &S_t^t - S_t^{t+1} = y_t^t + y_t^{t+1} \geq 0 \text{ for } t = 1, 2, \dots \end{aligned}$$

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where  $p^t \in R^{\lambda}$  is the vector of commodity prices in period  $t$ ;  $q^t$  is the resource price (both in present value, normalized on  $q^1 = 1$ ),  $S_t^i$  is the resource stock bought by agent  $t$  in period  $t$ ; and  $S_t^{i+1}$  is the stock sold by agent  $t$  in period  $t+1$ .

The solution to (1.1), if it exists, yields the demand function of agent  $t$  given by

$$f_0(p^1, q^1, \omega_0^i, S_0) = f_0^i = (x_0^i, -S_0^i) \in R^{\lambda+1} \text{ for } t=0$$

$$\begin{aligned} f_t(p^t, q^t, p^{t+1}, \omega_t^i, \omega^{t+1}) &= (f_t^i; f_t^{i+1}) \\ &= (x_t^i, S_t^i; x_t^{i+1}, -S_t^{i+1}) \in R^{2\lambda+2} \text{ for } t=1, 2, \dots \end{aligned}$$

and a price sequence  $p = \{(p^t, q^t)\}_{t=1}^{\infty}$  is an equilibrium price sequence if

$$(1.2) \quad f_t^{i+1} + f_{t+1}^{i+1} = (\omega_t^{i+1} + \omega_{t+1}^{i+1}, 0) \text{ for } t=0, 1, \dots$$

**1.1. Remark.** If agents are non-satiated in at least one good, then utility is strictly increasing in lifetime "wealth." By a simple equilibrium argument, any equilibrium price sequence must then have  $q^1 = q^2, \dots, = q$ . We shall thus restrict consideration to those price sequences for which  $q^t = 1$  for  $t=1, 2, \dots$

### The Theorem and Preliminary Discussion

**2.1. Theorem.** If agents have differentiable, monotone, strictly quasi-concave utility functions, the upper contour set of any  $x_t \gg 0$  is a proper subset of  $R^{2\lambda+2}$  (resp.  $R^{\lambda+1}$ ) (c.f. Debreu [3] and Arrow and Hahn [1]) and agents' commodity endowments are strictly positive in both periods of life and uniformly bounded, then equilibrium exists, resources will never be exhausted and each equilibrium allocation is fully Pareto optimal.

Proof of existence of equilibrium requires only minor modification of the strategy used for Proposition (3.11) of [2]. Since the hypotheses above encompass standard assumptions on utility functions (cf. Arrow and Hahn [1]), they insure that the constrained optimization problem (1.1) is well defined and that agents' demand functions exist and are continuous in lifetime wealth,  $p^t \omega_t + p^{t+1} \omega_t^{t+1}$  (resp.  $p^t \omega_0^i + q S_0$ ). Moreover, the solution to (1.1) is an interior maximum, i.e., demand is strictly positive for positive wealth and bounded prices. These two properties of any equilibrium must be verified and are the subject of the following two lemmas. The conclusion of the following is somewhat stronger than its analogue in [2] due to the existence of exhaustible resources.

**2.2 Lemma.** Under the hypotheses of the theorem, if  $\{(p^t, q)\}_{t=1}^{\infty}$  is a strictly positive equilibrium price sequence, then  $p^t$  is bounded for all  $t$  and  $\lim_{t \rightarrow \infty} \|p^t\| = 0$ .

**Proof.** Our proof is adapted from the proof of Lemma (3.4) of [2] differing only due to our more general setting. Consider first the market behavior of agent zero. His choices are restricted to the compact set  $\{x_0 \mid x_0 \leq (\omega_0^i + \omega_0^i, S_0), u_0(x_0) \geq u_0(\omega_0^i, S_0)\}$  and in equilibrium the normalized gradient of the utility function is

$$\nabla_n u_0(x_0) = \left( \frac{\partial u_0(x_0)^{-1}}{\partial y_0^i} \right) \quad \nabla u_0(x_0) = \frac{1}{q^1} (p^1, q) = (p^1, 1)$$

and since the mapping  $\nabla_n u_0: x_0 \rightarrow R^{\lambda+1}$  is continuous (cf. (1), p. 101),  $p^1$  is bounded. In particular, if commodity prices are not bounded, agent zero chooses

$y_0^i = S_0$  which cannot be an equilibrium since  $S_i^1 \geq y_0^i > 0$  for  $p^1 \geq 0$ . That is, if the increase in commodity prices is sufficiently large, agent zero will maximize utility by consuming the entire resource stock. This cannot be an equilibrium since with positive income agent one must demand a strictly positive amount of the exhaustible resource.

Now, consider the behavior of any agent  $t (t \geq 1)$  facing bounded (and positive) prices in his first period of life and having second-period endowment  $(\omega_t^{t+1}, S_t^t - y_t^t) \gg 0$ . For fixed first-period consumption, his choices are restricted to the compact set  $\{x_t | (x_t^{t+1}, y_t^{t+1}) < \omega_t^{t+1} + \omega_t^{t+1}, S_t^t - y_t^t \gg 0, u_t(x_t) \geq u_t(x_t^t, y_t^t; \omega_t^{t+1}, S_t^t - y_t^t)\}$  and in equilibrium

$$\nabla_n u_t(x_t) = (\partial u_t(x_t) / \partial y_t^t)^{-1} \nabla u_t(x_t) = (p^t, q; p^{t+1}, q)$$

and since the mapping  $\nabla_n u_t: x_t \rightarrow R^2 \times R^{+2}$  is continuous,  $p^{t+1}$  is bounded. As above, the argument implies that if  $p^{t+1}$  is unbounded, then  $S_t^{t+1} = 0$  which cannot be an equilibrium.

Moreover, since our assumptions on utility functions imply that  $y_t^t, y_t^{t+1} > 0$  (so long as either  $p^t$  or  $p^{t+1}$  is not zero) the sequence  $\{S_t^t\}_{t=0}^\infty$  is decreasing and  $\lim_{t \rightarrow \infty} S_t^t = 0$ . Because  $y_t^t \leq S_t^t$ ,  $\lim_{t \rightarrow \infty} y_t^t = 0$  so our assumptions on utility functions

imply  $\lim_{t \rightarrow \infty} \partial u_t(x_t) / \partial y_t^t = \infty$  for any feasible allocation and hence  $\lim_{t \rightarrow \infty} \|p^t\| = 0$ .

As in Balasko and Shell, the bounds imposed on prices by lemma (2.2) are independent of  $t$  and allow us to restrict attention to a given compact set of price sequences. We next verify that each agent's lifetime wealth is strictly positive. Intuitively, this occurs because each agent possesses commodities in his first period of life which contemporaneous agents require and have the resources to purchase.

**2.3 Lemma.** Under the hypotheses of the theorem, agents are resource related and equilibrium prices, if they exist, are strictly positive.

**Proof.** Consider any period  $t+1, t=0, 1, \dots$ . By construction, the initial allocation of agent  $t+1$  is  $(\omega_{t+1}^{t+1}, 0)$ ,  $\omega_{t+1}^{t+1} \gg 0$ , and that of agent  $t$  is  $(\omega_t^{t+1}, S_t^t - y_t^t) \gg 0$ . By hypothesis, any strictly positive allocation is strictly preferred by agent  $t+1$  and any allocation  $(x_t^{t+1}, S_t^t - y_t^t)$  with  $x_t^{t+1} \gg \omega_t^{t+1}$  is strictly preferred by agent  $t$ . Hence, agents are resource related and  $q^1 > 0$  implies  $p^{t+1} > 0$  (cf. [1], p. 104).

Since the upper contour set of each strictly positive allocation is a proper subset of  $R^2 \times R^{+2}$  (resp.  $R \times R^{+2}$ ) by hypothesis and wealth of each agent is strictly positive, if the price of any commodity is zero its excess demand is infinite and that price cannot be an equilibrium.

### Proof of the Theorem

Lemmas (2.2) and (2.3) allow us to apply Lemmas (3.8), (3.9) and Proposition (3.10) of [2], the only necessary modification being to replace the closure equation of Lemma (3.9) by

$$h_t^{t+1} = (\omega_t^{t+1}, S_t^{t+1})$$

Thus, equilibrium exists for the economy described in Section 1.

Now, suppose that  $\{(p^t, q)\}_{t=1}^{\infty}$  is an equilibrium price sequence generating demand by agent  $t$  which exhausts the resource stock in period  $t+1$ , i.e., that  $S_t^t > 0$  and  $S_t^{t+1} = 0$ . From Lemma (2.3) we know that agents are resource related and the lifetime wealth of agent  $t+1$  is strictly positive. However, for positive wealth and bounded prices demand by agent  $t+1$  is strictly positive; in particular, his resource demand is  $S_{t+1}^{t+1} > 0$ . Thus, we have excess demand  $(S_{t+1}^{t+1} - S_t^{t+1}) > 0$  for resources in period  $t+1$  and  $q > 0$  so  $\{(p^t, q)\}_{t=1}^{\infty}$  cannot be an equilibrium price sequence. Thus, in equilibrium, resources will never be exhausted.

Finally, to verify that each equilibrium allocation is Pareto optimal, let  $\{h_t\}_{t=1}^{\infty}$  be a sequence of Pareto improving transfers as in Definitions (5.1) and (5.2) of [2] with  $h_t = (a_t, b_t)$  where  $a_t$  is a vector of commodity transfers,  $b_t$  a transfer of resources. An argument identical to Lemma (5.5) implies that for  $t = 1, 2, \dots$

$$(3.1) \quad p^{t+1} a_{t+1}^{t+1} + q b_{t+1}^{t+1} \leq p^t a_t^t + q b_t^t \leq \dots \leq p^1 a_1^1 + q b_1^1 \leq 0$$

with strict inequality for  $t \geq t_0$  where  $t_0$  is the first agent for whom the transfer yields strictly greater utility. However, since  $b_t^t \leq s_t^t$ ,  $\lim_{t \rightarrow \infty} s_t^t = 0$ ,  $a_t^t$  is bounded and  $\lim_{t \rightarrow \infty} |p^t| = 0$ , we have

$$\lim_{t \rightarrow \infty} [p^t a_t^t + q^t b_t^t] = 0$$

in contradiction to (3.1). Thus, no Pareto improving transfer sequence exists.

### Conclusion

One generally expects that introduction of new factors, like exhaustible resources, into a model will substantially complicate its analysis. The remarkable result of this paper is that existence of such resources actually simplifies welfare analysis of the overlapping generations economy by insuring Pareto optimality of equilibrium allocations under much weaker conditions than those employed by Balasko and Shell. In retrospect, this is not entirely surprising, for existence of a truly essential exhaustible resource, together with boundedness of commodity endowments, implies both that long-run growth rates are non-positive and that long-run interest rates are strictly positive, thus providing the crucial link between equilibrium and optimality. Equally remarkable is the conclusion that if agents are resource related, an essential resource will never be exhausted; if equilibrium exists, "doomsday" will never occur.

We have established, in this paper, some basic existence and welfare properties of the overlapping generations economy with exhaustible natural resources. Beyond that, however, we have found that this model is an extremely fortuitous setting for study of resource allocation problems. There remain many unresolved problems concerning the economics of exhaustible resources, among them the roles of taxation, regulation, uncertainty and exploration. A more fully developed overlapping generations model seems to hold considerable promise for clarification of many of these difficult issues.

## REFERENCES

1. Arrow, K. and F. Hahn, *General Competitive Analysis*, San Francisco: Holden Day, 1971.
2. Balasko, Y. and K. Shell, "The Overlapping Generation Model I: The Case of Pure Exchange Without Money," *J. Econ. Theory* 23 (1980), 281-306.
3. Debreu, G. "Smooth Preferences," *Econometrica* 4 (1972), 604-620.
4. Samuelson, P. "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," *J. Pol. Economy* 66 (1958), 477-482.