

OPTIMUM LOCATION AND THE THEORY OF PRODUCTION: AN EXTENSION

*Abbas Mirakhor and A. Khalili**

In a paper in 1974, Khalili, et al [4], determined the condition for cost minimizing optimum production location for the case of Weber's locational triangle and the effect of changes in the level of output on the optimum location under conditions of perfect competition.

The purpose of this paper is (1) to derive profit maximizing conditions for the case of Weber's locational triangle when the firm is operating in imperfect input and output markets and (2) to determine the effects of income, output transport rate, elasticities of input supply curves on the optimum production location using comparative static analysis.

The "location problem" of the firm can be posed as follows: Assume a one-plant firm which is buying its inputs and selling its output in imperfectly competitive markets and which is interested in finding the optimum production location, uses two transportable inputs, M_1 and M_2 and supplies its single final product to a consumption center M_3 . The triangle in Figure 1 depicts this problem.

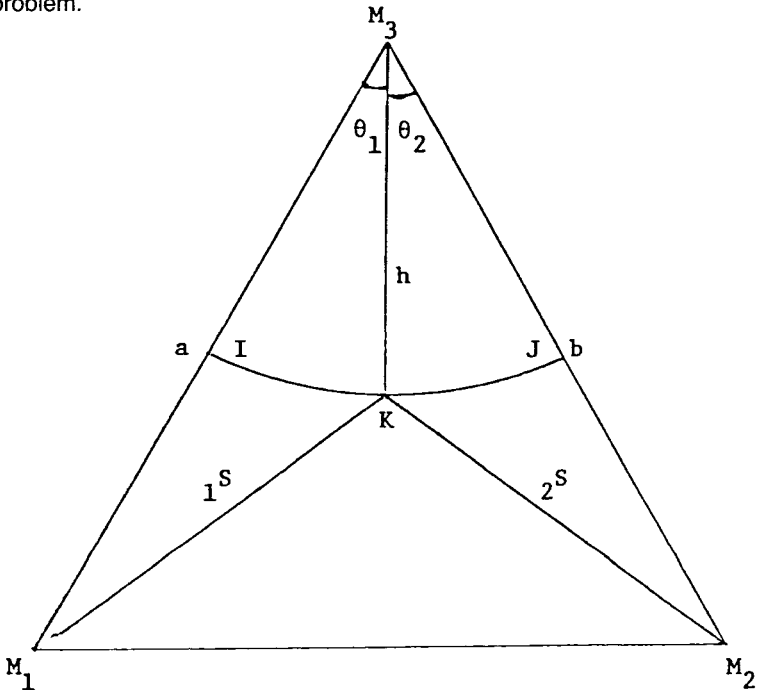


Figure 1.

* Florida Institute of Technology.

Mathematically the problem is to Maximize

$$\text{Maximize } \pi = R - C = P_0 F(M_1, M_2) - ({}_1P + r_{11}S)M_1 - ({}_2P + r_{22}S)M_2 - r_0 h F(M_1, M_2)$$

where, $P_0 = g(F, Y)$ is the price of the product, ${}_iP = {}_iP(M_i)$, $i = 1, 2$ is the price of input i and $F(M_1, M_2)$ is the production function. The distance from the production location to the sources M_1 and M_2 and to the market are ${}_1S$, ${}_2S$ and h respectively. The distances ${}_1S$ and ${}_2S$ could be expressed as

$${}_1S = \sqrt{a^2 + h^2 - 2ah\cos\theta}$$

$${}_2S = \sqrt{b^2 + h^2 - 2bh\cos(\theta - \theta_1)}$$

where a , b , and $\theta = \theta_1 + \theta_2$ are given. θ_1 and h are the polar coordinates of the production location point. r_0 is the transport rate of the final product to M_3 , r_1 and r_2 are transport rates of M_1 and M_2 respectively. The first order conditions are:

$$(1) \frac{\partial \pi}{\partial M_1} = g_1 F_1 F + F_1 g - {}_1P_1 M_1 - ({}_1P + r_{11}S) - r_0 h F_1 = \left\{ g \left(1 + \frac{F}{g} g_1 \right) - r_0 h \right\} F_1 - {}_1P' \left(1 + \frac{M_1}{{}_1P'} {}_1P'_1 \right) = 0$$

$$(2) \frac{\partial \pi}{\partial M_2} = g_1 F_2 F + F_2 g - {}_2P_2 M_2 - ({}_2P + r_{22}S) - r_0 h F_2 = \left\{ g \left(1 + \frac{F}{g} g_1 \right) - r_0 h \right\} F_2 - {}_2P' \left(1 + \frac{M_2}{{}_2P'} {}_2P'_2 \right) = 0$$

$$(3) \frac{\partial \pi}{\partial \theta_1} = -r_1 S_{\theta_1} M_1 - r_2 S_{\theta_1} M_2 = 0$$

$$(4) \frac{\partial \pi}{\partial h} = -r_1 S_h M_1 - r_2 S_h M_2 - r_0 F = 0$$

where P_j , $i = 1, 2$, $j = 1, 2$, F_1 , F_2 , ${}_1S_{\theta_1}$, ${}_2S_{\theta_1}$, ${}_1S_h$, ${}_2S_h$ are first partial derivatives, and $P'_i = P_i + r_{i1}S$; $P'_2 = P_2 + r_{22}S$.

Conditions (1) and (2) state that: net marginal revenue product (net of per unit output transport cost) for each of the two inputs must equal their respective marginal expense.

The total differential of the first order conditions are:

$$(5) (F_{11}B_1 + F_{12} \frac{\partial B_1}{\partial F} - {}_1P_{11}M_1 - 2{}_1P_1) dM_1 + (F_{12}B + F_1 F_2 \frac{\partial B_1}{\partial F}) dM_2 - r_{11} S_{\theta_1} d\theta_1 - (r_{11} S_h + r_0 F_1) dh = -B_2 F_1 dY + h F_1 dr_0 + {}_1S dr_1 - 2g_1 F_1 dF$$

$$(6) (F_{12}B_1 + F_1F_2 \frac{\partial B_1}{\partial F}) dM_1 + (F_{22}B_1 + F_2^2 \frac{\partial B_1}{\partial F} - {}_2P_{22}M_2 - {}_2P_2) dM_2 - r_{22}S_{\theta_1} d\theta_1 - (r_{22}S_h + r_oF_2) dh = -B_2F_2 dY + hF_2 dr_o + {}_2S dr_2 - 2g_1F_2 dF$$

$$(7) -r_{11}S_{\theta_1} dM_1 - r_{22}S_{\theta_1} dM_2 - C_{\theta_1\theta_1} d\theta_1 - C_{\theta_1h} dh = {}_1S_{\theta_1} M_1 dr_1 + {}_2S_{\theta_1} M_2 dr_2$$

$$(8) -(r_{11}S_h + r_oF_1) dM_1 - (r_{22}S_h + r_oF_2) dM_2 - C_{\theta_1h} d\theta_1 - C_{hh} dh = F dr_o + {}_1S_h M_1 dr_1 + {}_2S_h M_2 dr_2$$

$$\text{Let: } A_{11} = F_{11}B_1 + F_1^2 \frac{\partial B_1}{\partial F} - {}_1P_{11}M_1 - {}_2P_1$$

$$A_{12} = F_{12}B_1 + F_1F_2 \frac{\partial B_1}{\partial F}$$

$$A_{22} = F_{22}B_1 + F_2^2 \frac{\partial B_1}{\partial F} - {}_2P_{22}M_2 - {}_2P_2$$

where $B_1 = g + gF - r_o h$, $B_2 = g_{12}F + g_2$, $B_3 = 2g_1 + Fg_{11}$, $g < 0$ and ${}_1P_1, {}_2P_2$ are positive.

Then equations (5) - (8) become:

$$(9) A_{11} dM_1 + A_{12} dM_2 - r_{11}S_{\theta_1} d\theta_1 - (r_{11}S_h + r_oF_1) dh = -B_2F_1 dY + hF_1 dr_o + {}_1S dr_1 - 2g_1F_1 dF$$

$$(10) A_{12} dM_1 + A_{22} dM_2 - r_{22}S_{\theta_1} d\theta_1 - (r_{22}S_h + r_oF_2) dh = -B_2F_2 dY + hF_2 dr_o + {}_2S dr_2 - 2g_1F_2 dF$$

$$(11) -r_{11}S_{\theta_1} dM_1 - r_{22}S_{\theta_1} dM_2 - C_{\theta_1\theta_1} d\theta_1 - C_{\theta_1h} dh = {}_1S_{\theta_1} M_1 dr_1 + {}_2S_{\theta_1} M_2 dr_2$$

$$(12) -(r_{11}S_h + r_oF_1) dM_1 - (r_{22}S_h + r_oF_2) dM_2 - C_{\theta_1h} d\theta_1 - C_{hh} dh = F dr_o + {}_1S_h M_1 dr_1 + {}_2S_h M_2 dr_2$$

where:

$$C_{\theta_1\theta_1} = r_{11}M_1 {}_1S_{\theta_1\theta_1} + r_{22}M_2 {}_2S_{\theta_1\theta_1}$$

$$C_{\theta_1h} = r_{11}M_1 {}_1S_{\theta_1h} + r_{22}M_2 {}_2S_{\theta_1h}$$

$$C_{hh} = r_{11}M_1 {}_1S_{hh} + r_{22}M_2 {}_2S_{hh}$$

{see equations (15) - (17) of [4]}.

The second order condition requires that the principle minors of the relevant Hessian determinant alternate in sign.

Proposition 1: Assuming h is a positive constant, θ_1 is a variable ($\theta_1 < \theta$) and the firm's marginal revenue is an increasing function of income, then the firm's production location is independent of the level of income, and output transport rate, if and only if the expansion path is linear.

Proof: Using the system of equations (5)-(7) and holding h constant we have:

$$\frac{\partial \theta_1}{\partial Y} = \frac{1}{D^*} \begin{vmatrix} A_{11} & A_{12} & -B_2 F_1 \\ A_{12} & A_{22} & -B_2 F_2 \\ -r_{11} S_{\theta_1} & -r_{22} S_{\theta_1} & 0 \\ M_1 A_{11} + M_2 A_{12} & M_2 A_{12} & F_1 \\ M_1 A_{12} + M_2 A_{22} & M_2 A_{22} & F_2 \\ 0 & -r_{22} M_2 S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{-B_2}{D^* M_1 M_2}$$

where D^* is the relevant bordered Hessian (see [4]).

Expanding and simplifying this determinant, we obtain:

$$\frac{\partial \theta_1}{\partial Y} = \frac{-r_{22} S_{\theta_1} B_2}{D^* M_1} \{F_2(M_1 A_{11} + M_2 A_{12}) - F_1(M_1 A_{12} + M_2 A_{22})\}$$

$$= \frac{-r_{22} S_{\theta_1} B_2}{D^* M_1} N''$$

Since $r_{22} S_{\theta_1} < 0$, $D^* < 0$, and $B_2 > 0$, $\frac{\partial \theta_1}{\partial Y} = 0$ if and only if N'' is zero.

However, $N'' = 0$ if and only if the expansion path is linear through origin (see Appendix A).

Similarly

$$\frac{\partial \theta_1}{\partial r_o} = \frac{h}{D^* M_1 M_2} \begin{vmatrix} M_1 A_{12} + M_2 A_{12} & M_2 A_{12} & F_1 \\ M_1 A_{12} + M_2 A_{22} & M_2 A_{22} & F_2 \\ 0 & -r_{22} M_2 S_{\theta_1} & 0 \end{vmatrix} = 0$$

if and only if the expansion path is linear.

Proposition 2: If h is constant and greater than zero, $\theta_1 < \theta$, and the firm's marginal revenue is an increasing function of income the firm's optimum location would swing along the arc IJ (see Figure 1) towards $M_1(M_2)$ if and only if $M_1(M_2)$ increases relative to $M_2(M_1)$ along the expansion path as the level of income increases.

Proof: From the last proposition, one determines the sign of $\frac{\partial \theta_1}{\partial Y}$ to be

opposite that of N'' . Therefore, when $N'' > 0$ (< 0), $\frac{\partial \theta_1}{\partial Y} < 0$ (> 0) and the

firm's optimal location would move towards $M_1(M_2)$. But, $N'' > 0$ (< 0) if and only if $M_1(M_2)$ is used more relative to $M_1(M_2)$ along the expansion path (see Appendix B) as the level of income increases.

Proposition 3: Assuming h is constant and greater than zero, $\theta_1 < \theta$, the firm's optimum production location would swing along the arc IJ toward $M_1(M_2)$

if and only if $M_1(M_2)$ increases relative to $M_2(M_1)$ along the expansion path, as the output transport rate decreases.

Proof: Using Cramer's rule, from the system of equations (9) - (11), we find:

$$\frac{\partial \theta_1}{\partial r_o} = \frac{1}{D^*} \begin{vmatrix} A_{11} & A_{12} & hF_1 \\ A_{12} & A_{22} & hF_2 \\ -r_1 {}_1S_{\theta_1} & -r_2 {}_2S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{h}{D^* M_1 M_2} \begin{vmatrix} M_1 A_{12} + M_2 A_{12} & M_2 A_{12} & F_1 \\ M_1 A_{12} + M_2 A_{22} & M_2 A_{11} & F_2 \\ 0 & -r_2 M_2 {}_2S_{\theta_1} & 0 \end{vmatrix}$$

$$= \frac{r_2 h {}_2S_{\theta_1}}{D^* M_1} \{F_2(M_1 A_{11} + M_2 A_{12}) - F_1(M_1 A_{12} + M_2 A_{22})\} = \frac{r_2 h {}_2S_{\theta_1}}{D^* M_1} N''$$

Since $r_2 {}_2S_{\theta_1} < 0$ and $D^* < 0$, the sign of $\frac{\partial \theta_1}{\partial r_o}$ is the same as that of N'' .

Therefore, when $N'' > 0$ (< 0), $\frac{\partial \theta_1}{\partial r_o} > 0$ (< 0) and the firm's optimal location would move towards $M_1(M_2)$ as r_o decreases. But $N'' > 0$ (< 0) if and only if $M_1(M_2)$ is used more relative to $M_2(M_1)$ along the expansion path (see Appendix B).

Proposition 4: If both θ_1 and h are variables, and the firm's marginal revenue is an increasing function of income, then the production function is independent of the level of income if the production function is linearly homogeneous and the marginal expense elasticities of the input supply curves are equal.

Proof: Using the system of equations (5) - (8), we obtain:

$$\frac{\partial h}{\partial Y} = \frac{1}{D} \begin{vmatrix} A_{11} & A_{12} & -r_1 {}_1S_{\theta_1} & -B_2 F_1 \\ A_{12} & A_{22} & -r_2 {}_2S_{\theta_1} & -B_2 F_2 \\ -r_1 {}_1S_{\theta_1} & -r_2 {}_2S_{\theta_1} & -C_{\theta_1 \theta_1} & 0 \\ -(r_{11} S_h + r_o F_1) & -(r_{22} S_h + r_o F_2) & -C_{\theta_1 h} & 0 \end{vmatrix}$$

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2 M_2^2} \begin{vmatrix} M_1^2 A_{11} & M_1 M_2 A_{12} & -r_1 M_1 {}_1S_{\theta_1} & -B_2 M_1 F_1 \\ M_1 M_2 A_{12} & M_2^2 A_{22} & -r_2 M_2 {}_2S_{\theta_1} & -B_2 M_2 F_2 \\ -r_1 M_1 {}_1S_{\theta_1} & -r_2 M_2 {}_2S_{\theta_1} & -C_{\theta_1 \theta_1} & 0 \\ -(r_1 M_1 {}_1S_h + r_o M_1 F_1) & -(r_2 M_2 {}_2S_h + r_o M_2 F_2) & -C_{\theta_1 h} & 0 \end{vmatrix}$$

Let: $E_{11} = A_{11} M_1^2 + 2A_{12} M_1 M_2 + M_2^2 A_{22}$
 $E_{12} = M_1 M_2 A_{12} + M_2^2 A_{22}$
 $E_{22} = M_2^2 A_{22}$

Then:

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2 M_2^2} \begin{vmatrix} E_{11} & E_{12} & 0 & -B_2 n F \\ E_{12} & E_{22} & -r_2 M_2 {}_2 S_{\theta_1} & 0 \\ 0 & -r_2 M_2 {}_2 S_{\theta_1} & -C_{\theta_1, \theta_1} & 0 \\ -r_0 F (n-1) & -(r_2 M_2 {}_2 S_h + r_0 M_2 F_2) - C_{\theta_1, h} & & 0 \end{vmatrix}$$

Letting $n = 1$, multiplying the first row by $\frac{-M_2 F_2}{F}$ and adding it to the second we get:

$$\frac{\partial h}{\partial Y} = \frac{1}{DM_1^2 M_2^2} \begin{vmatrix} E_{11} & E_{12} & 0 & B_2 F \\ E_{12} - \frac{M_2 F_2}{F} E_{11} & E_{12} - \frac{M_2 F_2}{F} E_{12} & -r_2 M_2 {}_2 S_{\theta_1} & 0 \\ 0 & -r_2 M_2 {}_2 S_{\theta_1} & -C_{\theta_1, \theta_1} & 0 \\ 0 & -(r_2 M_2 {}_2 S_h + r_0 M_2 F_2) - C_{\theta_1, h} & & 0 \end{vmatrix}$$

$$\frac{\partial h}{\partial Y} = \frac{-2B_2(M_1 F_2 {}_1 P_1 - M_2 F_1 {}_2 P_2) \{r_2 M_2 {}_2 S_{\theta_1} C_{\theta_1, h} - (r_2 M_2 {}_2 S_h + r_0 M_2 F_2) C_{\theta_1, \theta_1}\}}{DM_1 M_2}$$

However, $E'_{12} = FE_{12} - M_2 F_2 E_{11} = M_1 M_2 (-M_2 F_1 B'_2 + M_1 F_2 B'_1)$ for $n = 1$, and if $\varepsilon_2 = \frac{M_2/dM_2}{C'_2/dC'_2} = \varepsilon_1 = \frac{M_1/dM_1}{C'_1/dC'_1}$, then $E'_{12} = 0$ (see Appendix C).

Therefore, $\frac{\partial h}{\partial Y} = 0$ for $n = 1$, and $\varepsilon_1 = \varepsilon_2$.

Similarly

$$\frac{\partial \theta_1}{\partial Y} = \frac{-B(M_1 F_2 {}_1 P_1 - M_2 F_1 {}_2 P_2) \{r_2 M_2 {}_2 S_{\theta_1} C_{h, h} + (r_2 M_2 {}_2 S_h + r_0 M_2 F_2) C'_{\theta_1, h}\}}{DM_1^2 M_2} = 0$$

for $n = 1$, and $\varepsilon_1 = \varepsilon_2$

APPENDIX A

$$\begin{aligned}
 (1) \quad N'' &= F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22}) \\
 M_1A_{11} + M_2A_{12} &= Q_4B_1 + Q_3F_1B_3 - {}_1P_{11}M_1^2 - 2M_{11}P_1 \\
 M_1A_{12} + M_2A_{22} &= Q_2B_1 + Q_3F_2B_3 - {}_2P_{22}M_2^2 - 2M_{22}P_2 \\
 F_2(M_1A_{11} + M_2A_{12}) - F_1(M_1A_{12} + M_2A_{22}) &= F_2Q_4B_1 + Q_3F_1F_2B_3 - \\
 &F_2({}_1P_{11}M_1^2 + 2M_{11}P_1) - F_1Q_2B_1 - Q_3F_1F_2B_3 + 2M_{22}P_2 = 0 \\
 &= F_2B_1Q_4 - F_1B_1Q_2 - F_2M_1({}_1P_{11}M_1 + 2{}_1P_1) \\
 &+ F_1M_2({}_2P_{22}M_2 + 2{}_2P_2) = 0 \\
 \text{where } Q_2 &= M_1F_{12} + M_2F_{22}, Q_3 = M_1F_1 + M_2F_2, Q_4 = M_1F_{11} + M_2F_{12}, \\
 B_1 &= g + Fg_1 - r_0h, \text{ and } B_3 = Fg_{11} + 2g_1.
 \end{aligned}$$

From the first order conditions, we have:

$$(2) \quad F_1B_1 = {}_1P' \left(1 + \frac{M_1}{{}_1P'} {}_1P'_1 \right) = C'_1$$

$$(3) \quad F_2B_1 = {}_2P' \left(1 + \frac{M_2}{{}_2P'} {}_2P'_2 \right) = C'_2$$

$$\text{Also, } \frac{dC'_1}{dM_1} = {}_1P_{11}M_1 + 2{}_1P_1 \text{ and } \frac{dC'_2}{dM_2} = {}_2P_{22}M_2 + 2{}_2P_2$$

Therefore, equation (1) becomes

$$C'_2(M_1F_{11} + M_2F_{12}) - C'_1(M_1F_{12} + M_2F_{22}) - F_2M_1 \frac{dC'_1}{dM_1} + F_1M_2 \frac{dC'_2}{dM_2} = 0$$

or

$$M_1(C'_2F_{11} - C'_1F_{12} - F_2 \frac{dC'_1}{dM_1}) - M_2(C'_1F_{22} - C'_1F_{22} - F_1 \frac{dC'_2}{dM_2}) = 0$$

or

$$(4) \quad \frac{M_1}{M_2} = \frac{C'_1F_{22} - C'_2F_{12} - F \frac{dC'_2}{dM_2}}{C'_2F_{11} - C'_1F_{12} - F \frac{dC'_1}{dM_1}}$$

From the first order conditions, we have:

$$(5) \quad \frac{F_1}{F_2} = \frac{C'_1}{C'_2} \text{ or } H(M_1, M_2) = \frac{C'_2F_1}{C'_1F_2} = 1$$

Differentiating H partially with respect to M_1 and M_2 , we get:

$$(6) \quad \frac{\partial H}{\partial M_1} = \frac{C_2' F_{11} \cdot C_1' F_2 - (C_1' F_{21} + F_2 \frac{dC_1'}{dM_1}) C_2' F_1}{(C_1' F_2)^2}$$

$$(7) \quad \frac{\partial H}{\partial M_2} = \frac{(C_2' F_{12} + F_1 \frac{dC_2'}{dM_2}) C_1' F_2 - C_1' F_{22} C_2' F_1}{(C_1' F_2)^2}$$

From (6) and (7)

$$(8) \quad \frac{\frac{\partial H}{\partial M_2}}{\frac{\partial H}{\partial M_1}} = \frac{(C_2' F_{12} + F_1 \frac{dC_2'}{dM_2}) C_1' F_2 - C_1' F_{22} C_2' F_1}{C_2' F_{11} - C_1' F_2 - (C_1' F_{21} + F_2 \frac{dC_1'}{dM_1}) C_2' F_1}$$

From the implicit function theorem, we have:

$$(9) \quad \frac{\frac{\partial H}{\partial M_2}}{\frac{\partial H}{\partial M_1}} = - \frac{dM_1}{dM_2}$$

Therefore,

$$(10) \quad \frac{(C_2' F_{12} + F_1 \frac{dC_2'}{dM_2}) C_1' F_2 - C_1' F_{22} C_2' F_1}{C_2' F_{11} - C_1' F_2 - (C_1' F_{21} + F_2 \frac{dC_1'}{dM_1}) C_2' F_1} = - \frac{dM_1}{dM_2}$$

From the first order conditions, we have:

$$C_2' F_1 = C_1' F_2$$

Therefore, equation (10) become:

$$(11) \quad \frac{(C_2' F_{12} + F_1 \frac{dC_2'}{dM_2}) C_1' F_2 - C_1' F_{22} C_1' F_2}{C_2' F_{11} \cdot C_1' F_1 - (C_1' F_{21} + F_2 \frac{dC_1'}{dM_1}) C_1' F_2} = - \frac{dM_1}{dM_2}$$

Simplifying (11), we get:

$$(12) \quad \frac{C_1'F_{22} = C_2'F_{12} - F_1 \frac{dC_2'}{dM_2}}{C_2'F_{11} - C_1'F_{21} - F_2 \frac{dC_1'}{dM_1}}$$

From (4) and (12) we get:

$$(13) \quad \frac{dM_1}{dM_2} = \frac{M_1}{M_2}$$

Equation (13) is that of a linear expansion path.

APPENDIX B

From Appendix A, we have

$$(14) \quad N'' = M_1(C_2'F_{11} - C_1'F_{12} - F_2 \frac{dC_1'}{dM_1}) - M_2(C_1'F_{22} - C_2'F_{12} - F_1 \frac{dC_2'}{dM_2})$$

For a firm (buying and selling in imperfect markets) using two inputs, only one of the inputs could be inferior, i.e., when M_2 is inferior, $C_2'F_{11} - C_1'F_{12} - F_2 \frac{dC_1'}{dM_1} > 0$, M_1 is superior and $C_1'F_{22} - C_2'F_{12} - F_1 \frac{dC_2'}{dM_2} < 0$. Therefore, $N'' > 0$

if and only if

$$(15) \quad \frac{M_2}{M_1} > \frac{C_2'F_{11} - C_1'F_{12} - F_2 \frac{dC_1'}{dM_1}}{C_1'F_{22} - C_2'F_{12} - F_1 \frac{dC_2'}{dM_2}}$$

From equation (12) of Appendix A, it follows that

$$(16) \quad \frac{M_2}{M_1} > \frac{dM_2}{dM_1}$$

Equation (16) implies that M_1 increases relative to M_2 along the expansion path. Similarly, in the event M_2 is superior and M_1 is inferior, $N'' < 0$ if and only if

$$(17) \quad \frac{M_2}{M_1} < \frac{dM_2}{dM_1}$$

which implies that the firm's location will move towards M_2 if and only if M_2 is used more intensively along the expansion path. When both M_1 and M_2 are superior, $N'' > 0$ if and only if the condition (16) holds. The converse is true when condition (17) holds.

APPENDIX C

$$\begin{aligned}
 (18) \quad E_{11} &= M_1^2 A_{11} + 2M_1M_2A_{12} + M_2^2 A_{22} \\
 M_1^2 A_{11} &= M_1^2 gF_{11} + 2M_1^2 g_1F_1^2 + M_1^2 g_{11}F_1^2 F + M_1^2 F_{11}Fg_1 - r_oM_1^2 hF_{11} \\
 2M_1M_2A_{12} &= 2M_1M_2gF_{12} + 4M_1M_2g_1F_1F_2 + 2M_1M_2F_1F_2Fg_{11} + \\
 &\quad 2M_1M_2g_1FF_{12} - 2r_oM_1M_2hF_{12} \\
 M_2^2 A_{22} &= M_2^2 gF_{22} + 2M_2^2 g_1F_2^2 + M_2^2 g_{11}FF_2^2 + M_2^2 Fg_1F_{22} - r_oM_2^2 hF_{22} \\
 E_{11} &= g(M_1^2 F_{11} + 2M_1M_2F_{12} + M_2^2 F_{22}) + 2g_1(M_1^2 F_1^2 + 2M_1M_2F_1F_2 + \\
 &\quad M_2^2 F_2^2 + g_{11}F(M_1^2 F_1^2 + 2M_1M_2F_1F_2 + M_2^2 F_2^2)) + g_1F(M_1^2 F_{11} + \\
 &\quad 2M_1M_2F_{12} + M_2^2 F_{22}) - r_o h(M_1^2 F_{11} + 2M_1M_2F_{12} + M_2^2 F_{22}) \\
 E_{11} &= n(n-1)Fg + 2g_1n^2F^2 + g_{11}n^2F^3 + n(n-1)g_1F^2 - n(n-1)Fr_o h \\
 &= n(n-1)Fg(1 + \frac{F}{P_o}g_1) + n^2F^2(2g_1 + g_{11}F) - n(n-1)Fr_o h \\
 &= n(n-1)F\{g + g_1F - r_o h\} + n^2F^2 \{2g_1 + Fg_{11}\} \\
 \text{For } n = 1, E_{11} &= F^2(2g_1 + Fg_{11}) = F^2B_3
 \end{aligned}$$

$$\begin{aligned}
 (19) \quad E_{12} &= M_1M_2A_{12} + M_2^2 A_{22} \\
 M_1M_2A_{12} &= M_1M_2gF_{12} + 2M_1M_2g_1F_1F_2 + M_1M_2F_1F_2Fg_{11} + M_1M_2g_1FF_{12} \\
 &\quad - r_oM_1M_2hF_{12} \\
 M_2^2 A_{22} &= M_2^2 gF_{22} + 2M_2^2 g_1F_2^2 + M_2^2 g_{11}FF_2^2 + M_2^2 Fg_1F_{22} - r_oM_2^2 hF_{22} \\
 E_{12} - (n-1)M_2F_2g + 2nM_2F_2Fg_1 + nM_2F_2F^2g_{11} + (n-1)M_2F_2Fg_1 - \\
 &\quad (n-1)r_oM_2hF_2 \\
 \text{For } n = 1, E_{12} &= 2M_2F_2Fg_1 + M_2F_2F^2g_{11} = M_2F_2FB_3
 \end{aligned}$$

$$\begin{aligned}
 (20) \quad E_{12}F - E_{11}M_2F_2 &= 2M_2F_2F^2g_1 + M_2F_2F^3g_{11} - 2M_2F_2F^2g_1 - \\
 M_2F_2F^3g_{11} &= 0
 \end{aligned}$$

Since the input markets are imperfect, then for $n = 1$

$$\begin{aligned}
 (21) \quad E'_{12} &= E_{12}F - E_{11}M_2F_2 = M_1M_2(-M_2F_1B'_2 - M_1F_2B'_1) = 0 \\
 \frac{M_2}{M_1} &= \left(\frac{F_2}{F_1}\right)\left(\frac{B'_1}{B'_2}\right) = \left(\frac{F_2}{F_1}\right)\left(\frac{dC'_1/dM_1}{dC'_2/dM_2}\right)
 \end{aligned}$$

$$\text{where } C'_1 = {}_1P'(1 + \frac{M_1}{{}_1P} {}_1P_1')$$

$$C'_2 = {}_2P'(1 + \frac{M_2}{{}_2P} {}_2P_2')$$

But from the first order conditions, we have

$$C'_1 = F_1 B_1$$

$$C'_2 = F_2 B_1$$

or

$$\frac{C'_2}{C'_1} = \frac{F_2}{F_1}$$

Therefore, relation (21) becomes

$$(22) \quad \frac{M_2}{M_1} = \frac{C'_2}{C'_1} \left(\frac{dC'_1/dM_1}{dC'_2/dM_2} \right)$$

Relation (22) could be written as

$$\frac{M_2/dM_2}{C'_2/dC'_2} = \frac{M_1/dM_1}{C'_1/dC'_1}$$

Therefore, if the marginal expense elasticities of input supply curves are equal, then $E'_{12} = 0$.

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